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Frequency-Domain Karhunen–Loeve Method and Its Application to Linear Dynamic Systems

Taehyoun Kim*
Boeing Commercial Airplane Group, Seattle, Washington 98124-2207

For the first time, the Karhunen–Loeve (KL) procedure is derived in the frequency domain as a tool for calculating eigenmodes of linear systems. The new derivation is based on the discrete Fourier transform representation of a time average of profile energy including a proper system response and a profile function. Taking the variational problem posed as such with respect to the profile function leads to an eigenformulation in the frequency domain. Choice of a system response for efficient KL mode calculation and construction of reduced-order systems using the KL eigenmodes are also discussed. To demonstrate the method, both mechanical and fluid dynamic models are considered. The method is equally useful in extracting eigenmodes of an experimentally generated database.

Nomenclature

c = wing chord length
$\boldsymbol{F}$ = snapshot matrix as defined in Eq. (13)
$\mathcal{F}$ = Fourier operator
$\mathcal{G}_{Ik}$ = impulse response for the kth input
$\mathcal{G}_{Sk}$ = step response for the kth input
j $\equiv \sqrt{-1}$
K = number of inputs or the kernel function defined as in Eq. (9)
L = dimension of state vector $\boldsymbol{y}$
M = number of frequency samples
N = number of time samples
$\boldsymbol{p}_R$ = $(R \times 1)$ generalized coordinate vector
R = number of chosen Karhunen–Loeve (KL) modes
t = time
$\boldsymbol{u}$ = $(K \times 1)$ input vector
$\mathcal{U}$ = Fourier transform of $\boldsymbol{u}$
V_∞ = freestream velocity
$\boldsymbol{v}_i$ = right eigenvector of Eq. (25)
$\boldsymbol{w}_i^T$ = left eigenvector of Eq. (25)
$\boldsymbol{x}$ = position vector ($\equiv \lfloor x_1 \;\; x_2 \;\; x_3 \rfloor^T$)
y = system response in continuous space domain
$\boldsymbol{y}$ = $(L \times 1)$ state vector
$\mathcal{Y}$ = frequency response vector or Fourier transform of $\boldsymbol{y}$
$\boldsymbol{y}_{SC}$ = static correction vector
α = eigenvector of Eq. (12)
$\bar{\lambda}$ = eigenvalue of Eq. (8) or Eq. (12)
μ_i = eigenvalue of Eq. (25)
$\boldsymbol{\xi}$ = dummy position vector, identical to $\lfloor \xi_1 \;\; \xi_2 \;\; \xi_3 \rfloor^T$
τ = reduced time, $t(V_\infty/0.5c)$
$\boldsymbol{\Phi}_R$ = $(L \times R)$ KL modal matrix
ϕ = KL eigenfunction
ω = frequency

I. Introduction

ALTHOUGH eigenfunctions, or eigenmodes, of simple linear systems are well known and used in typical modal analyses for extracting characteristics of low-frequency dynamics, the eigenvalue problem can be extremely difficult for more general, complex problems. For structural dynamics problems where undamped, free vibrational modes are sought, the large-scale eigenvalue problem may be solved with the help of powerful modern computers. However, computational fluid dynamics models usually produce large, sparse, nonsymmetric matrices that require even greater amounts of storage and computation time.[1] Obtaining the eigenmodes of the fluid systems may further be hampered by their increased sensitivity in the complex variable domain.

Recently, Karhunen–Loeve (KL) decomposition has been used in numerous applications to produce new sets of eigenmodes for dynamic and fluid–structure interaction applications.[2–6] There are several advantages of using the KL method over conventional eigenanalysis. First and foremost from an application point of view, the KL procedure uses the so-called snapshot method, where the problem of obtaining eigenmodes of a large system reduces to solving eigenmodes of matrices of order of only 10^2 (Refs. 2 and 4). Second, the method always produces real, optimal modes regardless of the damping characteristics of the system under consideration. Third, the method is a direct response approach that does not require a dynamic model describing the system. Hence, it can be applicable to both analytical and experimental models. Last, by solving a linear system together with its adjoint system, it may be possible to reconstruct eigenmodes of the original system.[3]

In this paper, a new KL procedure is derived in the frequency domain. The derivation is based on the discrete Fourier transform representation of a time average of profile energy with a given system response and a profile function. The variational problem is considered, and an eigenformulation is derived in the frequency domain. Also included are discussions of choosing a system response for best KL mode estimation and the construction of reduced-order systems based on the KL eigenmodes. For illustration, optimal sets of the KL eigenmodes are obtained for two different dynamic systems. They are a mechanical mass–spring–damper system with 12 degrees of freedom and an unsteady vortex lattice model of a rectangular wing in incompressible subsonic flow. The new optimal eigenmodes are compared with the direct system eigenmodes, and system responses are reproduced by including a relatively small number of modes. It is shown that the new method is an efficient and practical procedure for determining optimal eigenmodes of frequency-based linear systems.

II. Derivation of the Frequency-Domain KL Method

The following formulation is equally applicable to an experimentally as well as a computationally generated database. In the first case, it is assumed that certain frequency responses are available from an experiment. The responses should be measured at a sufficient number of stations and frequencies such that they define a dynamic system, in that such a realization must be achievable through the inverse Fourier transform.

We seek a real function $\phi(\boldsymbol{x}, t)$ for which the following profile energy is a maximum:

$$J \equiv \langle (\phi, y)^2 \rangle \equiv \left\langle \left[\int_{\mathcal{R}} \phi(\boldsymbol{x}) y(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} \right]^2 \right\rangle \tag{1}$$

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*Principal Engineer, Aeroelastic Analysis, P.O. Box 3707, MS 67-HM. Member AIAA.

with the constraint

$$h \equiv (\phi, \phi) - \langle (y, y) \rangle = 0 \tag{2}$$

where $y(\boldsymbol{x}, t)$ represents the response of a linear system under consideration. Here $\phi(\boldsymbol{x})$ represents an empirical eigenfunction that is optimal in that it participates in the response to the largest extent while maintaining the average system energy as its mean square value. The time average $\langle\,\rangle$ is approximated over a finite period T. Given N samples at $t_1\,(= -T/2), t_2, \ldots, t_N\,(= T/2)$, with step sizes $\Delta t_1, \Delta t_2, \ldots, \Delta t_N$,

$$\langle (\phi, y)^2 \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (\phi, y)^2 \,\mathrm{d}t \simeq \frac{1}{T} \sum_{i=1}^{N} \left[\phi(\boldsymbol{x}), y^{(i)}(\boldsymbol{x})\right]^2 \Delta t_i \tag{3}$$

where $y^{(i)}(\boldsymbol{x}) \equiv y(\boldsymbol{x}, t_i)$.

A. Direct Method

The mean square energy of (ϕ, y) expressed by Eq. (3) can be converted to an integral in the frequency domain by Parseval's theorem as

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (\phi, y)^2 \,\mathrm{d}t = \lim_{T,\Omega\to\infty} \frac{1}{2\pi T} \int_{-\Omega}^{\Omega} \|\mathcal{F}[(\phi, y)]\|^2 \,\mathrm{d}\omega \tag{4}$$

The magnitude of the frequency response is obtained as

$$\|\mathcal{F}[(\phi, y)]\|^2 = (\phi, \mathcal{Y}^*)\,(\phi, \mathcal{Y}) \tag{5}$$

where $\mathcal{Y}(\boldsymbol{x}, \omega) \equiv \mathcal{F}[y(\boldsymbol{x}, t)]$ and the asterisk represents the complex conjugate. Assuming that sampled data are available at M frequencies, $\omega_1\,(= -\Omega), \omega_2, \ldots, \omega_M\,(= \Omega)$, with step sizes $\Delta\omega_1, \Delta\omega_2, \ldots, \Delta\omega_M$, Eq. (1) can be approximated as

$$J \simeq \frac{1}{2\pi T} \sum_{i=1}^{M} \left\|\mathcal{F}^{(i)}[(\phi, y)]\right\|^2 \Delta\omega_i \tag{6}$$

where $\mathcal{F}^{(i)}[(\phi, y)] \equiv \mathcal{F}[(\phi, y)]|_{\omega=\omega_i}$. The frequency range $(-\Omega, \Omega)$ sweeps both negative and positive values. With this convention, it is assumed that the positive frequencies $0 < \omega < \Omega$ correspond to $1 \le n \le M/2$, whereas negative ones $-\Omega < \omega < 0$ correspond to values $M/2 + 1 \le n \le M$. With the constraint (2), the functional to be maximized is

$$\hat{J} = J - \lambda h \tag{7}$$

Taking the variational of Eq. (7) and setting it equal to zero leads to

$$\int_{\mathcal{R}} K(\boldsymbol{x}, \boldsymbol{\xi}) \phi(\boldsymbol{\xi}) \,\mathrm{d}\boldsymbol{\xi} = \bar{\lambda} \phi(\boldsymbol{x}) \qquad (\bar{\lambda} \equiv 2\pi T \lambda) \tag{8}$$

where

$$K(\boldsymbol{x}, \boldsymbol{\xi}) \equiv \sum_{i=1}^{M} \mathcal{Y}^{*(i)}(\boldsymbol{x}) \mathcal{Y}^{(i)}(\boldsymbol{\xi}) \Delta\omega_i \tag{9}$$

and $\mathcal{Y}^{(i)}(\boldsymbol{x}) \equiv \mathcal{Y}(\boldsymbol{x}, \omega_i)$. As in the time-domain version, the kernel function K can be interpreted as the correlation of the two system responses, $\mathcal{Y}(\boldsymbol{x}, \omega)$ and $\mathcal{Y}(\boldsymbol{\xi}, \omega)$. It is noted that $K(\boldsymbol{x}, \boldsymbol{\xi})$ is real symmetric, and without loss of generality the eigenmodes $\phi(\boldsymbol{x})$ are real and orthonormal, i.e., $(\phi_i, \phi_j) = \delta_{ij}$.

B. Snapshot Method

The direct KL formulation has advantages for a system with a moderate spatial resolution. However, if the system has a considerably higher resolution, as is the case for large-scale simulations of complex systems, the snapshot method is preferable. A significant amount of size reduction can be achieved by assuming $\phi(\boldsymbol{x})$ is a linear combination of instantaneous snapshots.[2,4] In the frequency domain, this statement can be expressed as

$$\phi(\boldsymbol{x}) = \sum_{i=1}^{M} \alpha_i \mathcal{Y}^{(i)}(\boldsymbol{x}) \sqrt{\Delta\omega_i} \tag{10}$$

where α_i is a sequence of complex numbers weighted with the frequency steps. Substituting Eq. (10) into Eq. (8) yields

$$\sum_{j=1}^{M} \int_{\mathcal{R}} \mathcal{Y}^{*(i)}(\boldsymbol{x}) \mathcal{Y}^{(j)}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \sqrt{\Delta\omega_i} \sqrt{\Delta\omega_j} \alpha_j = \bar{\lambda} \alpha_i \qquad (i, j = 1, 2, \ldots, M) \tag{11}$$

Or in matrix form,

$$\Delta\boldsymbol{\omega}^{\frac{1}{2}} \boldsymbol{F} \Delta\boldsymbol{\omega}^{\frac{1}{2}} \boldsymbol{\alpha} = \bar{\lambda} \boldsymbol{\alpha} \tag{12}$$

where

$$F_{ij} \equiv \int_{\mathcal{R}} \mathcal{Y}^{*(i)}(\boldsymbol{x}) \mathcal{Y}^{(j)}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \tag{13}$$

$$\boldsymbol{\alpha} \equiv \lfloor \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_M \rfloor^T \tag{14}$$

$$\Delta\boldsymbol{\omega}^{\frac{1}{2}} \equiv \mathrm{diag}\left(\sqrt{\Delta\omega_i}\right) \tag{15}$$

Hence, the integral equation (8) is reduced to the eigenvalue equation of order M, Eq. (12). Note that $\boldsymbol{F}$ is Hermitian, i.e., $\boldsymbol{F} = (\boldsymbol{F}^*)^T$. As a result, all eigenvalues $\bar{\lambda}$ are real, and the eigenvectors $\boldsymbol{\alpha}_i$ are orthogonal. In addition, elements of each $\boldsymbol{\alpha}_i$ form complex conjugate pairs. Therefore, all of the system eigenmodes as obtained by Eq. (10) are real and orthonormal.

A general frequency response can now be approximated in a linear combination of the eigenfunctions ϕ_i as

$$\mathcal{Y}(\boldsymbol{x}, \omega) \simeq \sum_{i=1}^{R} \mathcal{A}_i(\omega) \phi_i(\boldsymbol{x}) \qquad (R \le M) \tag{16}$$

where, as a result of the orthonormality,

$$\mathcal{A}_i = (\phi_i, \mathcal{Y}) \tag{17}$$

Using the property of the kernel K, it can be shown that

$$\bar{\lambda}_i = \left\langle \|(\phi_i, \mathcal{Y})\|^2 \right\rangle = \left\langle \|\mathcal{A}_i\|^2 \right\rangle \tag{18}$$

That is, the eigenvalue of a KL mode is also a measure of how much the mode participates in generating the system response under question.

Except for the scaling factor $2\pi T$, the frequency-domain KL procedure given in either Eq. (8) or Eq. (12) precisely resembles the time-domain eigenvalue problem

$$\int_{\mathcal{R}} \sum_{i=1}^{N} y^{(i)}(\boldsymbol{x}) y^{(i)}(\boldsymbol{\xi}) \Delta t_i \phi(\boldsymbol{\xi}) \,\mathrm{d}\boldsymbol{\xi} = \lambda \phi(\boldsymbol{x}) \tag{19}$$

In fact, $\mathcal{A}_i(\omega)$ in Eq. (16) can be interpreted as the Fourier transforms of the time coefficients $a_i(t)$ that satisfy

$$y(\boldsymbol{x}, t) = \sum_{i=1}^{N} a_i(t) \phi_i(\boldsymbol{x}) \tag{20}$$

Consequently, the same set of eigenmodes would result from both versions of KL procedures provided that $y^{(i)}(\boldsymbol{x})$ and $\mathcal{Y}^{(i)}(\boldsymbol{x})$ constitute exact Fourier transform pairs.

III. Choice of System Response for KL Method

When applied to linear systems, the KL procedure poses the following fundamental question: Which system response should be used to calculate the best set of eigenmodes for all possible responses? The question arises because we are interested in not only obtaining the KL modes but also using them for general response problems. One aspect of the KL method not particularly suitable for this goal is that it is a noncausal procedure. That is, eigenmodes cannot be obtained before a system response is known a priori. In this section, some aspects of this issue will be explored, and two types system responses are provided for use in producing the KL eigenmodes.

To this end, some properties of a linear dynamic system will be examined. It is assumed that the system has a finite number of state variables L and is subjected to a finite number of inputs K. Defining $\mathcal{G}_{Ik}$ and $\mathcal{G}_{Sk}$ as the impulse and step response for the kth input U_k, the frequency response can be expressed as a linear combination of these responses:

$$\mathcal{Y}(\omega) = \sum_{k=1}^{K} \mathcal{G}_{Ik}(\omega) U_k(\omega) \tag{21}$$

$$= \sum_{k=1}^{K} j\omega \mathcal{G}_{Sk}(\omega) U_k(\omega) \tag{22}$$

The corresponding response in the time domain becomes, ignoring initial conditions,

$$\boldsymbol{y}(t) = \sum_{k=1}^{K} \int_0^t \boldsymbol{g}_{Ik}(t-\tau) u_k(\tau)\, \mathrm{d}\tau \tag{23}$$

$$= \sum_{k=1}^{K} \int_0^t \boldsymbol{g}_{Sk}(t-\tau) \frac{\partial u_k}{\partial \tau}(\tau)\, \mathrm{d}\tau \tag{24}$$

At this point, to understand how the inputs excite the characteristic nature of the system, we will use the notion of system eigenmodes. These are not to be confused with the KL modes and are, in fact, the solutions of the homogeneous part of the following mathematical realization of the system:

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y} + \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_K \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_K \end{Bmatrix} \tag{25}$$

With μ_i, $\boldsymbol{v}_i$, and $\boldsymbol{w}_i^T$ as the eigenvalues and the corresponding right and left eigenvectors of the system matrix $\boldsymbol{A}$, respectively, one can write

$$\mathcal{G}_{Ik} = \sum_{i=1}^{L} \beta_{ik} \frac{1}{j\omega - \mu_i} \boldsymbol{v}_i \tag{26}$$

where $\beta_{ik} \equiv \boldsymbol{w}_i^T \boldsymbol{b}_k$. The kth input will excite the ith system mode if and only if $\beta_{ik} \neq 0$. In a most general case, all system modes would participate in the system response, but the amount of a particular mode will be magnified or attenuated proportionally to $\|\beta_{ik}/(j\omega - \mu_i)\|$. One can then form groups of eigenvectors $V_1 \equiv [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{p1}]$, $V_2 \equiv [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{p2}], \ldots, V_K \equiv [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{pK}]$, for each $\boldsymbol{u}_k$ such that, for any selected eigenvector $\boldsymbol{v}_i$ and a given scalar number α_0, $\|\beta_{ik}/(j\omega - \mu_i)\| \geq \alpha_0$ within a given frequency range. Let us assume that the sum of these groups consists of total of Q eigenmodes as follows:

$$V_s \equiv [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_Q] \tag{27}$$

Assuming that the Q system eigenmodes (unknown for application) are the major ones of interest, the following procedures for the KL mode calculations can be established:

1) Obtain a series of KL mode sets from $\mathcal{G}_{Ik}$ or $\mathcal{G}_{Sk}$ $(k = 1, 2, \ldots, K)$ sequentially. These sets will span $V_1, V_2, \ldots, V_K$, respectively, and hence span V_s.

2) Obtain a single set of KL modes from

$$\sum_{k=1}^{K} w_k \mathcal{G}_{Ik} \qquad \text{or} \qquad \sum_{k=1}^{K} w_k \mathcal{G}_{Sk}$$

that will span V_s.

Of the two methods, the first is more complete but may be too costly if there are many inputs. It may also produce redundant KL modes. The second method is more economical, but one must either check whether all of the $\boldsymbol{b}_k$ are linearly independent, or if not, find a proper set of weights w_k such that

$$\sum_{k=1}^{K} w_k \boldsymbol{b}_k$$

include all of the independent $\boldsymbol{b}_k$. The conditions under which these requirements are met will depend on the specific nature of inputs, including their locations and directions. Furthermore, the range of frequency sampling may need to be adjusted accordingly. For a single-input system, however, the two methods yield the same results.

IV. Reduced-Order Dynamic System

In this section, construction of reduced-order systems using the KL modes is described. We seek forced responses of an L-dimensional linear dynamic system subjected to K inputs, described in the preceding section by Eq. (25),

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{B}\boldsymbol{u} \tag{28}$$

or in the frequency domain

$$\mathcal{Y}(\omega) = (j\omega \boldsymbol{I} - \boldsymbol{A})^{-1} \boldsymbol{B} \mathcal{U}(\omega) \tag{29}$$

When a model reduction is performed using a few eigenmodes, $\phi_1, \phi_2, \ldots, \phi_R$, where $R \leq M \ll L$, the full-order system Eq. (28) is projected onto the subdimensional space according to

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}_R \boldsymbol{p}_R(t)$$
$$\equiv [\phi_1 \quad \phi_2 \quad \cdots \quad \phi_R] \begin{Bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_R(t) \end{Bmatrix} \tag{30}$$

where p_i represent the generalized coordinates. For a better representation of the modal analysis, a static correction is added to account for the effects of the neglected modes, assuming that these modes participate in a quasistatic way.[1] The static correction is obtained as the difference between quasistatic responses of the full system and the reduced-order system:

$$\boldsymbol{y}_{\mathrm{SC}} \equiv \boldsymbol{y}_S - \boldsymbol{\Phi}_R \boldsymbol{p}_{RS} \tag{31}$$

After substituting Eq. (30) with the static correction (31) into Eq. (28) and premultiplying by $\boldsymbol{\Phi}_R^T$, we obtain the reduced-order system as

$$\dot{\boldsymbol{p}}_R = \boldsymbol{A}_R \boldsymbol{p}_R + \boldsymbol{B}_R \hat{\boldsymbol{u}} \tag{32}$$

$$\boldsymbol{y} = \boldsymbol{\Phi}_R \boldsymbol{p}_R + \boldsymbol{D}\hat{\boldsymbol{u}} \tag{33}$$

where

$$\boldsymbol{A}_R \equiv \boldsymbol{\Phi}_R^T \boldsymbol{A} \boldsymbol{\Phi}_R \tag{34}$$

$$\boldsymbol{B}_R \equiv \boldsymbol{\Phi}_R^T [(\boldsymbol{A}\boldsymbol{S} + \boldsymbol{I})\boldsymbol{B} \quad -\boldsymbol{S}\boldsymbol{B}] \tag{35}$$

$$\boldsymbol{S} \equiv -\boldsymbol{A}^{-1} + \boldsymbol{\Phi}_R \boldsymbol{A}_R^{-1} \boldsymbol{\Phi}_R^T \tag{36}$$

$$\boldsymbol{D} \equiv [\boldsymbol{S}\boldsymbol{B} \quad \boldsymbol{0}] \tag{37}$$

$$\hat{\boldsymbol{u}} \equiv \begin{Bmatrix} \boldsymbol{u} \\ \dot{\boldsymbol{u}} \end{Bmatrix} \tag{38}$$

We have used the orthonormality $\boldsymbol{\Phi}_R^T \boldsymbol{\Phi}_R = \boldsymbol{I}_R$. This procedure, known as Galerkin's method, is equivalent to minimizing the error associated with the approximation by constraining residual vector to be orthogonal to the subspace being used.

A case of special interest is when $R = M$. If the whole family of the KL modes is used, it can be shown by relations (10) and (16) that the system response is approximated as a linear combination of its solutions at a finite number of frequencies:

$$\mathcal{Y}(\boldsymbol{x}, \omega) \simeq \sum_{i=1}^{M} \mathcal{B}_i(\omega) \mathcal{Y}^{(i)}(\boldsymbol{x}) \tag{39}$$

provided that $[\mathcal{Y}^{(1)}(\boldsymbol{x}) \;\; \mathcal{Y}^{(2)}(\boldsymbol{x}) \;\; \cdots \;\; \mathcal{Y}^{(M)}(\boldsymbol{x})]$ has the full rank M. Indeed, if the system is evaluated at enough frequencies, such solutions must encompass all of the system eigenmodes within the chosen frequency range. Therefore, these responses can serve as basis functions.

Finally, some key observations are made about the role of the static correction. Equation (31) can be expressed in terms of linear combinations of zero frequency responses of the two systems as

$$y_{SC} \equiv ([y_{S1} \quad y_{S1} \quad \cdots \quad y_{SK}] - \Phi_R [p_{RS1} \quad p_{RS2} \quad \cdots \quad p_{RSK}])u \tag{40}$$

where y_{Si} and p_{RSi} are the static responses of the full- and reduced-order systems, respectively, for the ith unit input. It can be shown that, if the modal matrix Φ_R includes y_{Si}, then y_{SC} becomes identically zero, making the static correction inefficient. Hence, there exist special groups of KL modes for which the static correction is not required. As will be seen in examples to follow, if $\mathcal{G}_{Sk}$ are used sequentially for KL calculation, there exist modes from each set that correspond to the static responses of the system. Similarly, if a complete set of $\mathcal{G}_{Ik}$ is used, the general response will include the static responses at zero frequency according to Eq. (39). With these and similar exceptions, the static correction will generally improve the system response by accounting for quasistatic responses of the neglected KL modes. This was not clearly addressed in Ref. 5.

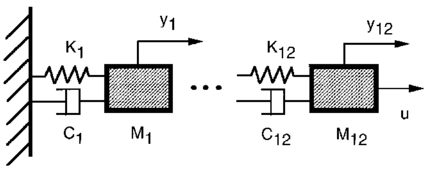

Fig. 1 Mechanical mass–spring–damper system with 12 degrees of freedom.

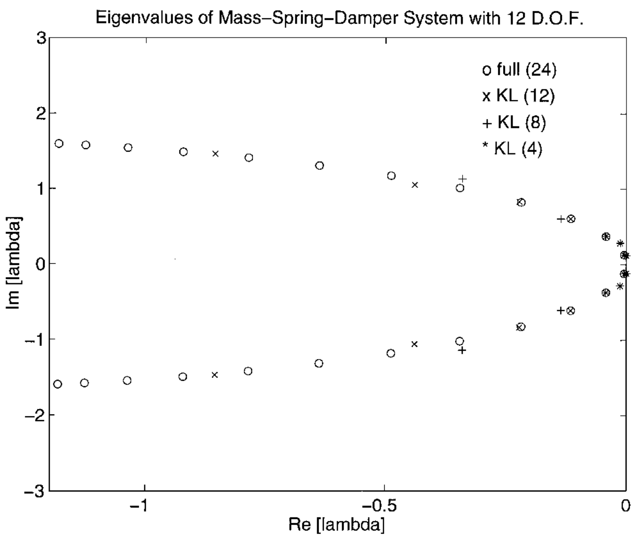

Fig. 2 Eigenvalues of mass–spring–damper system.

V. Numerical Results

In this section, the frequency-domain KL method is applied to two simple dynamic systems. For simplicity of demonstration, the examples are limited to low-order models with single inputs. With a few exceptions noted, static corrections were included in all of the results. It was found that, for the range of reduced-order models examined, the use of the snapshot method yielded KL modes almost identical to those of the direct method. Therefore, only the reduced-order models using the snapshot method will be reported.

The first example is a 12-degree-of-freedom mass–spring–damper system, where one unit consists of a mass attached to a spring and a viscous damper in parallel and where all units are connected to each other in series (Fig. 1). The left-hand base is fixed, and the entire system is driven by a force input at the last nodal point. Uniform mass, spring, and damper distributions were assumed with $m_i = 1.0$, $k_i = 1.0$, and $c_i = 0.6$, respectively. For KL mode calculation, a unit step input was applied at 21 frequencies uniformly sampled between 10^{-4} and 3.1888 rad/s. This produced a total of 42 KL eigenmodes. Figure 2 shows eigenvalues of reduced-order systems using the first 4, 8, and 12 KL modes, respectively, whose eigenvalues λ have been arranged in descending order. They are the eigenvalues of the homogeneous part, $\dot{p}_R = A_R p_R$. Also shown are the direct eigenvalues of the full-order system.

It can be seen that, beginning with the lowest ones, more system eigenmodes are captured as more KL modes are added. Figure 3 represents frequency responses of the reduced-order models at the 12th mass point. Because the step input was used for KL calculation, the first KL mode with the largest eigenvalue was indeed the static response of the system. Hence, adding static corrections did not improve the results. For comparison, Fig. 4 represents frequency responses of reduced-order models obtained by direct reduction using a few system eigenmodes v_i along with static corrections. It is noted that, in both Figs. 3 and 4, the reduced models accurately

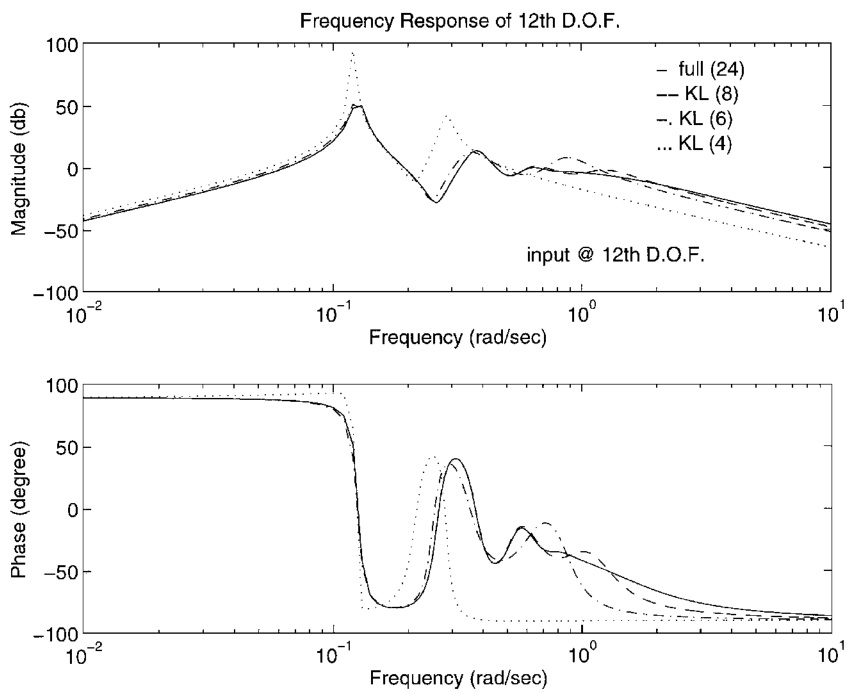

Fig. 3 Frequency responses of mechanical KL reduced models (based on step input).

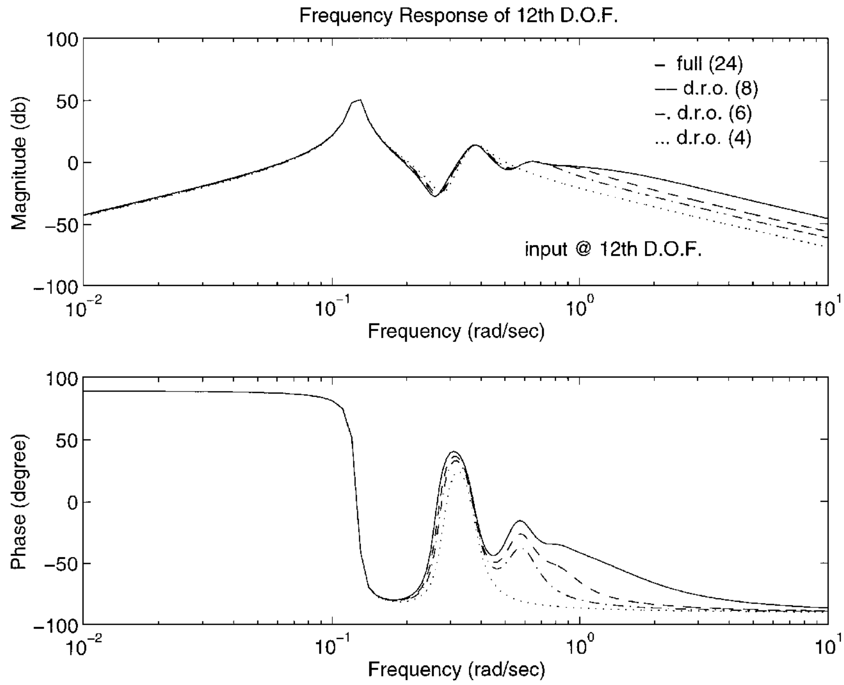

Fig. 4 Frequency responses of mechanical direct reduced models.

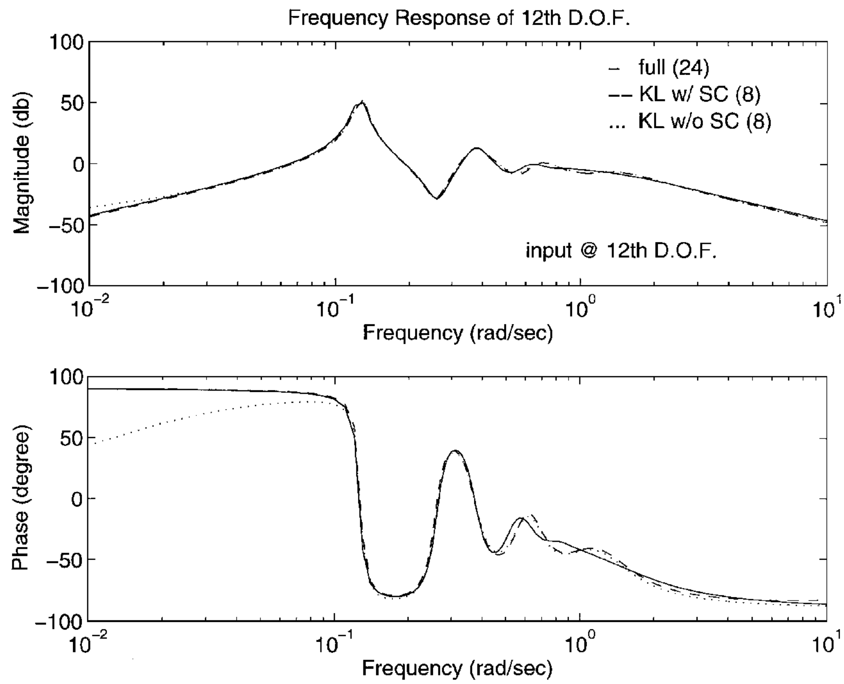

Fig. 5 Frequency responses of mechanical KL reduced models (based on impulse input).

capture the zero frequency gain of the original system. Comparing the two plots, it can be concluded that six KL modes are required for a good approximation that can be achieved by only four system eigenmodes. However, once the first six KL modes are included, the KL reduction gives slightly better models, particularly in the higher-frequency region.

To illustrate the importance of the static correction, frequency responses were regenerated based on unit impulse input (Fig. 5). The impulse input was applied at six sampling frequencies between 0 and 0.79719 rad/s, and the first eight KL modes were used with and without the static correction. Figure 5 clearly shows that it is necessary to include the static correction to capture the low-frequency response accurately. However, it was found that the static correction was not needed if the entire 12 KL modes were used.

The next example is a three-dimensional, unsteady vortex lattice model in subsonic incompressible flow (Fig. 6). The discrete-time vortex model was provided in Ref. 7, where eigenmodes were calculated directly from the system equation. Its continuous-time version of a reduced-order model was obtained and used for aeroservoelastic applications in Ref. 8. A rectangular wing of aspect ratio five has been discretized using rectangular vortex ring elements. Symmetry was assumed, and only half of the wing surface was analyzed. The wing chord and semispan were divided into 6 and 10 equal lengths, respectively. The wake was assumed to be five times longer than the wing chord. Hence, the size of $\boldsymbol{y}$ is (300×1).

For the KL mode calculation, a step pitch motion was applied. The air speed was assumed to be 57.2 m/s in all cases. The time step used in the preceding vortex model is $\Delta t = \Delta x / U_\infty$, where Δx is

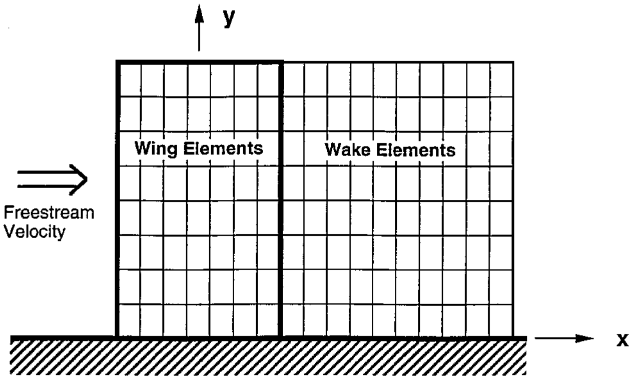

Fig. 6 Vortex lattice model of a finite aspect ratio wing.

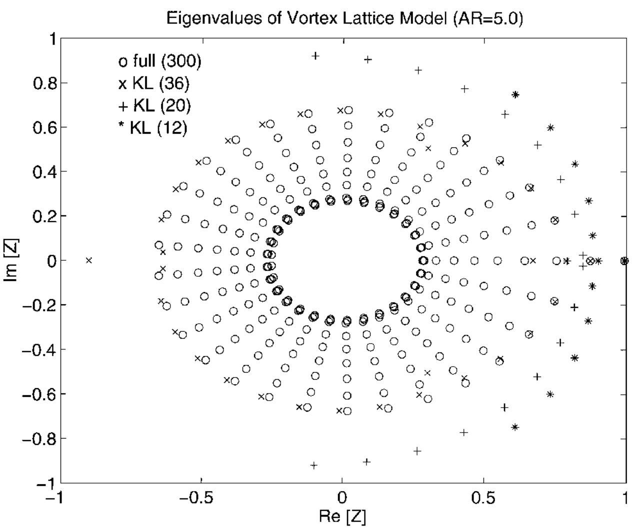

Fig. 7 Eigenvalues of vortex lattice model in *Z* domain.

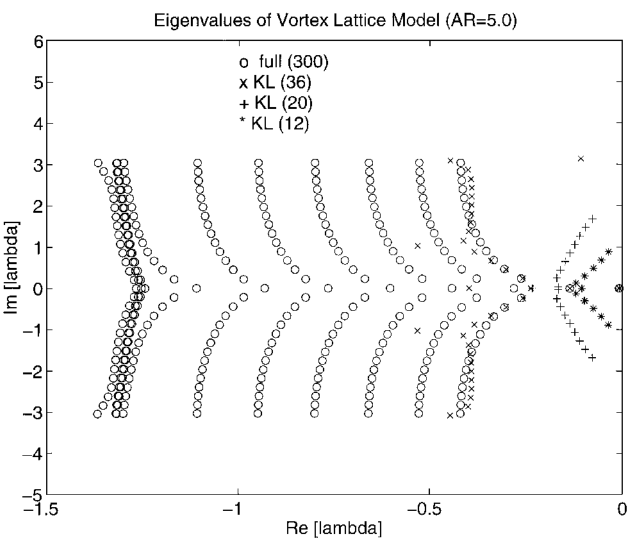

Fig. 8 Eigenvalues of vortex lattice model in *S* domain.

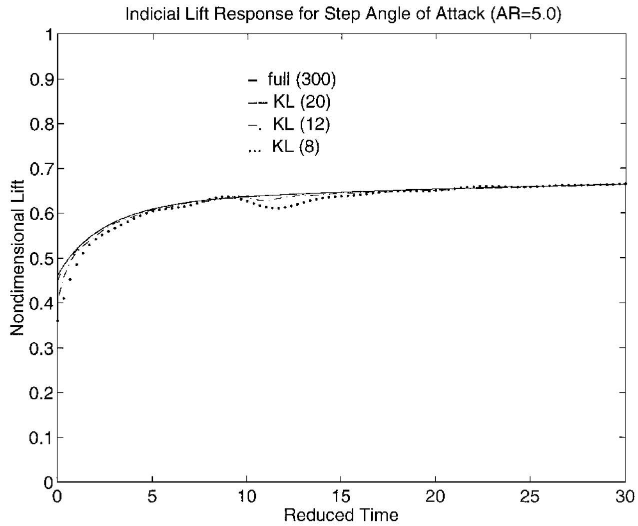

Fig. 9 Indicial lift growths for a step angle of attack.

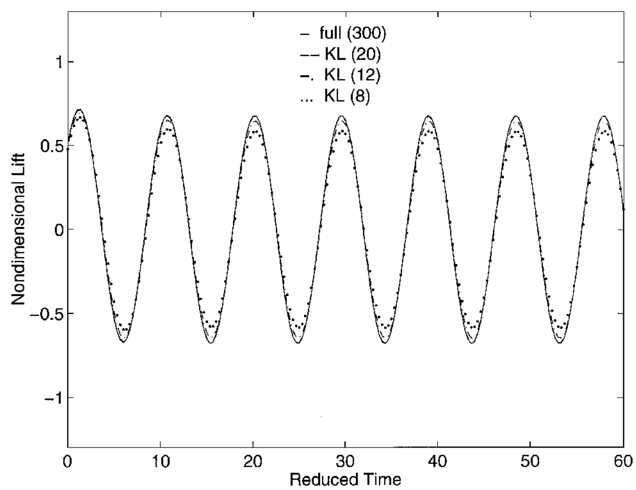

Fig. 10 Unsteady lift responses for a sinusoidal input $\sin(0.6661\tau)$.

the chord length of a vortex element, and the maximum allowable frequency is $\Omega = \pi/\Delta t$ according to the Nyquist sampling rule. A total of 52 equally distributed frequencies between 10^{-4} and Ω were used in the KL procedure.

Figures 7 and 8 show eigenvalue distributions of the reduced-order models in the Z ($= e^{s\Delta t}$) and S domains, respectively, using 12, 20, and 36 KL modes that correspond to the largest $\bar{\lambda}$ in descending order. Also shown in Figs. 7 and 8 are the direct eigenvalues of the full-order vortex model. Figures 7 and 8 clearly show that, as the number of KL modes is increased, the reduced-order model captures the first branch of the full-order system eigenmodes. However, it was found that, once the number is increased beyond 30, which is the number of chordwise nodes in the wake, the remaining KL modes do not capture the next branches well, indicating that the first eigenmode branch participates mostly in the step response.

In Fig. 9, indicial lift growths are generated using 8, 12, and 20 KL modes, respectively, by subjecting the entire wing to a step angle of attack. It is seen that including the first 20 KL modes reproduces the lift curve very accurately. In Fig. 10, several responses subject to a sinusoidal pitch motion, $\sin(0.6661\tau)$, are presented. Once again, including the 20 KL modes produces excellent results. As in the earlier example, static corrections were not required in both results.

VI. Concluding Remarks

A new KL procedure has been derived in the frequency domain. Construction of reduced-order models using the KL eigenmodes was also discussed. Using frequency responses generated from simple, low-order dynamic models, it was demonstrated that the present method is equally as useful as the time-domain KL method in extracting eigenmodes of linear systems. The new KL eigenmodes are real, orthogonal, and optimal, as are the time-domain-based KL modes. The KL modes in the present examples were calculated using single inputs. However, for more general response problems that involve multiple inputs, the choice of input for the KL method is not unique. This issue should be investigated further in the future. As stated in the Introduction, we expect the greatest benefits from the KL methods when solving large, complex linear systems, analytical or experimental. In that context, the methods have already been shown to be very efficient in several areas, including unsteady aerodynamics and aeroelasticity.

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References

[1]Dowell, E. H., Hall, K. C., and Romanowski, M. C., "Eigenmode Analysis in Unsteady Aerodynamics; Reduced-Order Models," *Applied Mechanics Review*, Vol. 50, No. 6, 1997, pp. 371–386.

[2]Sirovich, L., Kirby, M., and Winter, M., "An Eigenfunction Approach to Large Scale Transitional Structures in Jet Flow," *Physics of Fluids A*, Vol. 2, No. 2, 1990, pp. 127–136.

[3]Breuer, K. S., and Sirovich, L., "The Use of the Karhunen–Loeve Procedure for the Calculation of Linear Eigenfunctions," *Journal of Computational Physics*, Vol. 96, No. 2, 1991, pp. 277–296.

[4]Winter, M., Barber, T. J., Everson, R. M., and Sirovich, L., "Eigenfunction Analysis of Turbulent Mixing Phenomena," *AIAA Journal*, Vol. 30, No. 7, 1992, pp. 1681–1688.

[5]Romanowski, M. C., "Reduced-Order Unsteady Aerodynamic and Aeroelastic Models Using Karhunen–Loeve Eigenmodes," *Proceedings of the AIAA Symposium on Multidisciplinary Analysis and Optimization* (Bellevue, WA), AIAA, Reston, VA, 1996, pp. 7–13 (AIAA Paper 96-3981).

[6]Park, H. M., and Lee, M. W., "An Efficient Method of Solving the Navier–Stokes Equations for Flow Control," *International Journal for Numerical Methods in Engineering*, Vol. 41, No. 6, 1998, pp. 1133–1151.

[7]Hall, K. C., "Eigenanalysis of Unsteady Flows About Airfoils, Cascades, and Wings," *AIAA Journal*, Vol. 32, No. 12, 1994, pp. 2426–2432.

[8]Kim, T., Nam, C., and Kim, Y., "Reduced-Order Aeroservoelastic Model with an Unsteady Aerodynamic Eigenformulation," *AIAA Journal*, Vol. 35, No. 6, 1997, pp. 1087, 1088.

A. Berman
Associate Editor